Higher Dimensional Kerr–AdS Black Holes and the AdS/CFT Correspondence

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Abstract

Using the counterterm subtraction technique we calculate the stress–energy tensor, action, and other physical quantities for Kerr–AdS black holes in various dimensions. For Kerr–AdS$_5$ with both rotation parameters non–zero, we demonstrate that stress–energy tensor, in the zero mass parameter limit, is equal to the stress tensor of the weakly coupled four dimensional dual field theory. As a result, the total energy of the general Kerr–AdS$_5$ black hole at zero mass parameter, exactly matches the Casimir energy of the dual field theory. We show that at high temperature, the general Kerr–AdS$_5$ and perturbative field theory stress–energy tensors are equal, up to the usual factor of 3/4. We also use the counterterm technique to calculate the stress tensors and actions for Kerr–AdS$_6$, and Kerr–AdS$_7$ black holes, with one rotation parameter, and we display the results. We discuss the conformal anomalies of the field theories dual to the Kerr–AdS$_5$ and Kerr–AdS$_7$ spacetimes. In these two field theories, we show that the rotation parameters break conformal invariance but not scale invariance, a novel result for a non–trivial field theory. For Kerr–AdS$_7$ the conformal anomalies calculated on the gravity side and the dual (0,2) tensor multiplet theory are equal up to 4/7 factor. We expect that the Casimir energy of the free field theory is the same as the energy of the Kerr–AdS$_7$ black hole (with zero mass parameter), up to that factor.

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1   Introduction

There has been considerable interest in Anti de–Sitter (AdS) black holes since the discovery of the AdS/CFT correspondence[1], relating the physics of AdS in $n+1$ dimensions to a strongly coupled conformal (gauge) field theory (CFT) in one dimension fewer. This interest was triggered by Witten [2], who in formulating a precise statement of the duality, (see also ref.[3]) found the dual interpretation of the thermodynamical properties of Schwarzchild AdS black holes (Sch–AdS) in terms of the dual field theory at a non–zero temperature related to the mass of the black hole. An example of this remarkable connections is the interpretation of the Hawking–Page phase transition[4] from AdS to Sch–AdS at finite temperature, as terms of a transition between phases of the dual strongly coupled field theory.

Since then many new works have been presented on AdS black holes with the three simplest properties of mass, charge and rotation, in the context of the AdS/CFT correspondence. Reissner–Nordstrom AdS (RN–AdS) black holes were studied in this context in ref.[5]. There are again remarkable interpretations of the black holes' properties in terms of the dual gauge field theory. There is a beautiful family of phase transitions as a function of temperature and charge, (also temperature and potential). These are interpreted in the gauge theory as a family of transitions at finite temperature and a chemical potential for a global charge under the R–symmetry of the (broken) supersymmetry of the gauge theory. This latter is especially interesting, since for the appropriate choice of R–current, the chemical potential in the dual gauge theory would be a close cousin of the baryon density operator in QCD. Therefore quite remarkably, studies of such charged black holes might give a window on the phase structure of gauge theory (and perhaps ultimately, QCD) at finite temperature and density, which is of great experimental interest. Further work on this is of great interest.

Kerr–AdS black holes were studied for the first time in the AdS/CFT context in ref.[6], where the authors studied three, four and five dimensional Kerr–AdS black holes and some features of their dual field theories. Several important contributions have been subsequently presented on Kerr–AdS black holes [7]–[15], addressing the subject of AdS/CFT duality.

Asymptotically AdS black holes enjoy many interesting properties distinct from black holes which are asymptotically Minkowskian. For example, the presence of a negative cosmological constant, $\Lambda < 0$, allows more geometries for black holes horizons than in the $\Lambda = 0$ case. These horizons can be spherical, planar and hyperbolic surfaces. Another interesting property that is not shared with $\Lambda = 0$ solutions is that AdS black holes are thermodynamically stable, in the canonical ensemble, for certain ranges of their parameters. Schawrzchild–AdS black holes which are large compared with the scale, $l \sim |\Lambda|^{-1/2}$, set by the cosmological constant, have positive specific heat, which is roughly attributable[4] to the fact that AdS effectively acts as a box with reflecting walls supplied by its natural boundary at infinity. Kerr–AdS black holes

are also stable in this way, and again this is not the case for ordinary Kerr. Kerr–AdS solutions have Killing vectors which are rotating with respect to time–translation Killing vectors, and time–like everywhere. This allows thermal radiation to rotate with the black hole's angular velocity all the way to infinity. As a result, superradiance is not allowed, and the black hole is in thermal equilibrium with the thermal radiation around it [6, 11].

In this paper we present the results of continued studies of Kerr–AdS black holes, using the counterterm subtraction method recalled in the next section. We calculate stress–energy tensors as well as gravitational actions of Kerr–AdS$_{n+1}$ spaces, for $n = 4, 5$ and 6, where the five dimensional case is for the general solution with two rotation parameters, while we consider only the one–parameter solutions for higher dimensions.

In the general Kerr–AdS$_5$ case we find that the stress–energy tensor, for zero mass parameter, exactly matches that of the weakly coupled dual field theory (four dimensional large N $SU(N)$ gauge theory on a rotating spacetime –see below) at zero temperature. Consequently, the Casimir energies and conformal anomalies of the dual theory exactly match their Kerr–AdS$_5$ counterparts. This happens as a consequence of a non–renormalization theorem which protects anomaly coefficients from higher loop corrections, specifically for this dual field theory. In Kerr-AdS$_7$ the situation is a little different, since there is no non–renormalization theorem that protects "type-A" anomaly coefficients (reviewed below) from higher loop corrections. As a result, we find that the conformal anomalies on both sides of the correspondence (the dual field theory is the six dimensional "(0, 2)" tensor multiplet field theory at large N) are not exactly equal, (see ref.[16] for the field theory result) there is a factor of 4/7 difference between them, presumably corresponding to a renormalisation in going from weak to strong coupling. We believe that the same factor will show up in comparing a perturbative Casimir energy calculation to the dual energy computed for Kerr–AdS$_7$. We compute and display the latter here, although there is no field theory calculation on the market to compare to yet. In Kerr–AdS$_5$ and Kerr-AdS$_7$ cases we find, by examining the form of the trace of the energy–momentum tenor, that the rotation parameters break conformal but not scale invariance. We also argue that at high temperature, the dual stress–energy tensors are equal up to the usual 3/4 factor encountered in comparing such unprotected things across the weak/strong coupling divide[17].

We completed these calculations and announced them in ref.[18], where we used the stress tensors as examples of the scale/conformal invariance issue mentioned in the previous paragraph. We suppressed some of the computational details there, promising to display them later. This is the main purpose of this paper. We have learned that a recent paper has appeared with computations which overlap with some of those presented here[19].

2 Computing the Action and Stress Tensor

The gravity action on a region of spacetime $\mathcal{M}$, with boundary $\partial\mathcal{M}$ has the form[39],

$$I_{\text{bulk}} + I_{\text{surf}} = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+1}x\sqrt{-g}\left(R + \frac{n(n-1)}{l^2}\right) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^n x\sqrt{-h}\,K. \qquad (1)$$

The first term is the Einstein–Hilbert action with negative cosmological constant Λ, which defines a natural length scale as follows: $\Lambda = -n(n-1)/2l^2$. The second term is the Gibbons–Hawking boundary term. Here, h_{ab} is the boundary metric and K is the trace of the extrinsic curvature K^{ab} of the boundary. We of course wish to consider the case where $\mathcal{M}$ is a complete spacetime which is asymptotically AdS, therefore having infinite volume.

To deal with the divergences which appear in the gravitational action (arising from integrating over the infinite volume), we shall use the "counterterm subtraction" method[21]–[23]. The method regulates the action by the addition of certain boundary counterterms which depend upon the geometrical properties of the boundary of the spacetime. They are chosen to diverge at the boundary in such a way as to cancel the bulk divergences[21]–[24] (see also [25, 9, 26, 27]):

$$I_{\text{ct}} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^n x\sqrt{-h}\left[\frac{(n-1)}{l} - \frac{l\mathcal{R}}{2(n-2)} + \frac{l^3}{2(n-4)(n-2)^2}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{n}{4(n-1)}\mathcal{R}^2\right)\right]. \quad (2)$$

Here $\mathcal{R}$ and $\mathcal{R}_{ab}$ are the Ricci scalar and tensor for the boundary metric h. Using these counterterms one can construct a divergence–free stress–energy tensor from the total action $I = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}}$ by defining (see *e.g.* [28]):

$$T^{ab} = \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h_{ab}}. \qquad (3)$$

The metric restricted to the boundary, h_{ab}, diverges due to an infinite conformal factor, which will turn out to be r^2/l^2 in coordinates we will introduce later.

We will be interested in comparing quantities computed for $\mathcal{M}$ to a strongly coupled dual field theory, which resides on a spacetime with a metric conformal to h_{ab}.[1] We take the background metric upon which the dual field theory resides as

$$\gamma_{ab} = \lim_{r\to\infty} \frac{l^2}{r^2} h_{ab}. \qquad (4)$$

and so the dual strongly coupled field theory's stress–tensor, $\widehat{T}^{ab}$ [2], is related to the one in equation (3) by the rescaling[29]:

$$\sqrt{-\gamma}\,\gamma_{ab}\widehat{T}^{bc} = \lim_{r\to\infty} \sqrt{-h}\,h_{ab}T^{bc}. \qquad (5)$$

This amounts to multiplying all expressions for T^{ab} displayed later by $(r/l)^{n-2}$ before taking the limit $r\to\infty$.

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[1]Sometimes this is just referred to as "the theory on the boundary", which is only roughly true.

[2]The expression for $\widehat{T}^{ab}$ has been computed for arbitrary background manifold in ref.[31].

3 The General Kerr–AdS$_5$ Solution

The rotation group in $n+1$ dimensions is $SO(n)$. The number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is the integer part of $n/2$. This means that general Kerr–AdS$_5$ has two parameters. Its metric is given by [6]:

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 \\
&\quad + \frac{(1+r^2/l^2)}{r^2\rho^2}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2 \\
&\quad + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(bdt - \frac{(r^2+b^2)}{\Xi_b}d\psi\right)^2 ,
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\rho &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \\
\Xi_a &= 1 - a^2/l^2, \quad \Xi_b = 1 - b^2/l^2 \\
\Delta_r &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2/l^2) - 2MG, \\
\Delta_\theta &= 1 - a^2/l^2\cos^2\theta - b^2/l^2\sin^2\theta .
\end{aligned}
\tag{7}
$$

The inverse temperature, computed by requiring regularity of the Euclidean section, is:

$$
\beta = \frac{2\pi r_+(r_+{}^2+a^2)(r_+{}^2+b^2)l^2}{2r_+^6 + r_+^4(l^2+b^2+a^2) - a^2b^2l^2} .
\tag{8}
$$

The stress energy tensor components calculated for general Kerr–AdS$_5$ are considerably long expressions. We found that once we performed the procedures outlined in the previous section, the resulting stress–energy tensor, proposed to be that for the strongly coupled dual field theory, can be written in the following compact form:

$$
\widehat{T}_a^b = \frac{M}{8\pi l^3}\left[4u_au^b + \delta_a{}^b\right] - \frac{l^3}{64\pi G}\left[\frac{1}{12}\delta_a{}^bR^2 - R^{cd}R_{cad}{}^b\right],
\tag{9}
$$

where R, R_{ab} and R_{abcd} are the Ricci scalar, Ricci tensor and Riemann tensor of the background metric γ_{ab}. Here, $u^a = (1,0,0,0)$ is a unit time–like vector and M is the mass parameter. We shall discuss in more detail the above form of the stress–energy tensor by the end of the subsection 3.2, where there and in the next subsection we test whether this is indeed related to the stress tensor of the large N $SU(N)$ Yang–Mills theory.

The mass and angular momenta calculated from this tensor are given by:

$$
\mathcal{M} = \frac{\pi l^2}{96G\Xi_a\Xi_b}[7\Xi_a\Xi_b + \Xi_a{}^2 + \Xi_b{}^2 + 72GM/l^2] ,
\tag{10}
$$

and

$$\mathcal{J}_\phi = \frac{\pi M a}{2\Xi_a{}^2\Xi_b}, \quad \mathcal{J}_\psi = \frac{\pi M b}{2\Xi_b{}^2\Xi_a} \ . \tag{11}$$

The angular velocities on the horizon have the form:

$$\Omega_\phi = a\frac{\Xi_a}{r_+^2 + a^2}, \quad \Omega_\psi = b\frac{\Xi_b}{r_+^2 + b^2} \ . \tag{12}$$

The action is given by

$$
\begin{aligned}
I_5 \ = \ & -\frac{\pi\beta l^2}{96\Xi_a\Xi_b G}\left[12(r_+{}^2/l^2)(1-\Xi_a-\Xi_b)+\Xi_a{}^2+\Xi_b{}^2+\Xi_b\Xi_a+12r_+{}^4/l^4-2(a^4+b^4)/l^4\right.\\
& \left. -12(a^2b^2/l^4)(r_+{}^{-2}l^{-2}-1/3)-12\right] \ .
\end{aligned}
\tag{13}
$$

while the area of the horizon is

$$\mathcal{A} = \frac{2\pi^2 G(r_+{}^2+a^2)(r_+{}^2+b^2)}{r_+\Xi_a\Xi_b} \ . \tag{14}$$

We note that the above quantities satisfy the following thermodynamical relation for $n=4$

$$S = \beta\left(\mathcal{M}-\Omega_i\mathcal{J}_i\right)-I_{n+1} = \frac{\mathcal{A}}{4G} \ , \tag{15}$$

which is a highly non–trivial check of our calculations. (Note that the expressions for angular momenta (11) are different from those given in ref.[6], where there is a missing factor of $\Xi_a{}^{-1}$ or $\Xi_b{}^{-1}$ in their expressions. The angular momenta and the other quantities calculated in that reference therefore do not satisfy the above thermodynamical relation, but with the expressions presented here, they do.)

3.1 The Conformal Anomaly

The dual field theory is proposed to be strongly coupled $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang–Mills theory with the supersymmetry broken by the rotation and the finite temperature (mass of the black hole). The spacetime metric on which the dual field theory resides is found, using the definitions in section 2, to be:

$$ds^2 = \left[-dt^2 + \frac{2a\sin^2\theta}{\Xi_a}dtd\phi + \frac{2b\cos^2\theta}{\Xi_b}dtd\psi + l^2\frac{d\theta^2}{\Delta_\theta} + l^2\frac{\sin^2\theta}{\Xi_a}d\phi^2 + \frac{l^2\cos^2}{\Xi_b}\theta d\psi^2\right] \ . \tag{16}$$

It is important to note that this generalized rotating Einstein universe[6] is conformally flat. This will be of great use later.

Now we would like to compare some of the results we have found on the AdS side of the correspondence with their dual on the field theory side. It is difficult to calculate physical

quantities in the strongly coupled field theory in order to make the comparison directly. Sometimes, however, there is a non–renormalization theorem which protects anomaly coefficients from higher loop corrections.

When this is true, the Casimir energy and the anomaly coefficients for both strongly coupled and weakly coupled limits of the theory should be the same. These anomaly coefficients are protected because the trace of the stress–energy tensor and the divergence of the $SU(4)$ R–symmetry current are in the same supermultiplet, which is called anomaly multiplet. The divergence of the R–current is just the chiral anomaly, which is one loop exact by Adler–Bardeen theorem. (See *e.g.* refs.[32] for relevant discussions.)

Unfortunately, our treatment does not include any supersymmetry, as it is broken by the rotation, and therefore we should not expect that we have any right to protection from a non–renormalisation theorem. Fortunately, higher loop corrections (corrections beyond planar diagrams) are of order $O(1)$ and $O(\frac{1}{N})$ [33]–[36] and so will not change the one loop results as long as we study the large N limit, which is precisely where the duality holds (at least if we stay in pure supergravity). This is the main reason that the conformal anomaly and Casimir energy for weakly coupled fields will be exactly equal to their duals on the gravity side, we will see in the following.

The definition of the stress tensor for the dual field theory that we shall use for weak coupling is the one introduced by Brown and Cassidy [37]. They calculated the renormalized stress tensor for free fields (conformally coupled scalar, Weyl spinor and $U(1)$ gauge field). Their expression is given by:

$$< \widehat{T}_{ab}^s >= H_{ab}{}^{(4)} - \frac{1}{16\pi^2} \left[\frac{1}{9}\alpha^s H_{ab}{}^{(1)} + 2\delta^s H_{ab}{}^{(3)} \right] , \qquad (17)$$

where $H_a^{b(1)}$, $H_a^{b(3)}$, α^s and δ^s are defined in ref.[38]. (Note that they use spacetime indices (μ, ν) for the field theory and β^s for the coefficient of $H_a^{b(3)}$ while here we use (a, b) for the indices and δ^s for the coefficient.) If the expectation value is over vacuum state, $H_{ab}{}^{(4)}$ will be zero, since it is identified with the vev of the stress–energy tensor in Minkowski space. We will use this fact in section 3.2. The label $s\in\{0, 1/2, 1\}$ distinguishes the spin of the field for which the labelled coefficients α^s and δ^s are computed. We can now compare this to the the non–thermal (*i.e.,* the M–independent) part of the tensor which we have computed on the gravity side, inserting the spins appropriate to the content of the dual theory: a gauge field, six scalars and four Weyl spinors.

According to the AdS/CFT relation, one should define the Casimir energy for the field theory dual to the Kerr–AdS spacetime as the contribution to the total energy of the spacetime which is independent of the black hole's mass[23]. We find that the two energies exactly match

and they are given by:

$$\mathcal{E} = \frac{N^2}{48l\Xi_a\Xi_b}(7\Xi_a\Xi_b + \Xi_a{}^2 + \Xi_b{}^2) \ . \tag{18}$$

where we used the standard entry in the AdS/CFT dictionary[1]

$$\frac{1}{G} = \frac{2N^2}{\pi l^3}. \tag{19}$$

This Casimir energy agrees with the one calculated in ref.[12] for one parameter Kerr–AdS$_5$ case as $b \to 0$.

Furthermore the two stress–energy tensors at zero mass parameter limit, are equal. This is not surprising, since conformal anomaly fixes the whole stress tensor in conformally flat background, once the quantum state is determined[38].

Let us now investigate conformal anomaly for dual field theory,[2, 21] on the boundary. The general form of Weyl anomaly in arbitrary dimension n is given by (see *e.g.* the discussions in refs.[40, 16]):

$$T_a{}^a = c_0 E_n + \Sigma c_i I_i + \nabla_i J^i \tag{20}$$

where E_n is Euler density in n dimensions, I_i are invariants which contain the Weyl tensor and its derivatives, and the last term is a collection of total derivative terms that can be removed by adding suitable local couterterms.[3] The first type of term is called "type A", the second "type B", and the last "type D". It is important to note that the coefficients of all terms are regularisation scheme independent *except* the type D anomaly.

In the two non–trivial cases of spherical Kerr–AdS$_5$ and Kerr–AdS$_7$, the boundary metric is conformally flat which means that the Weyl invariant terms vanish. The only surviving non–trivial term is the Euler density which is locally a total derivative. The Euler density integral is a topological invariant which vanishes for the boundaries of Kerr–AdS$_5$ and Kerr–AdS$_7$ spaces. In *special cases* the Euler density is proportional to a combination of terms in in the type D anomaly. This doesn't make the type A anomaly a type D anomaly, since the coefficient c_0 is scheme independent. Attempting to add counterterms to remove the type A anomaly will change the anomaly coefficient c_0, which means that we have moved to a different theory, instead of merely changing scheme.

We find[18] that this special situation is just what happens for the Kerr–AdS$_5$ and Kerr–AdS$_7$ solutions. The conformal anomaly for two parameters Kerr–AdS$_5$ has the following form:

$$T_a{}^a = -\frac{(a^2 - b^2)N^2}{4\pi^2 l^6}\left[a^2/l^2\cos^2\theta(3\cos^2\theta - 2) + b^2/l^2(\cos^2\theta(-3\cos^2\theta + 4) - 1) - \cos 2\theta\right] \ , \tag{21}$$

---

[3]This term is the generalization of the familiar $\Box R$ in four dimensions.

which matches exactly the general form of the trace anomaly in four dimensions. In terms of the Euler density, it can be written simply as

$$T_a{}^a = c_0 E_n \tag{22}$$

where $c_0 = -N^2/\pi^2$. The integrated anomaly is zero as in the one–parameter case

$$\int dx^4 \sqrt{-\gamma} T_a{}^a = 0 \ . \tag{23}$$

Since the energy–momentum tensor is traceful, but the trace is an irremovable total derivative, we see that for arbitrary rotation parameters we have broken conformal invariance but preserved scale invariance. This is discussed at length in ref.[18]. It is interesting to note that when the rotation parameters are equal, the anomaly vanishes identically.

3.2 The Stress–Energy Tensor at High Temperature

We show in this section that at high temperature the stress–energy tensor we computed for the general Kerr-AdS$_5$ case matches that of the dual field theory, up to the usual 3/4 factor encountered in going from weak to strong coupling.

Let us start with the renormalized stress–energy tensor of Brown and Cassidy for the field theory given in equation (17). Replacing the vacuum expectation value by the ensemble average $<>_\beta$ one obtains the thermal stress–energy tensor:

$$< T_a{}^b(\gamma) >_\beta \;\; = \;\; \omega^{-4} < T_a{}^b(\eta) >_\beta \; - \frac{1}{16\pi^2} \left[\frac{1}{9} \alpha H^{(1)}{}_a{}^b + 2\delta H^{(3)}{}_a{}^b \right]. \tag{24}$$

Here γ_{ab} is the spacetime metric of the theory, η_{ab} is that of Minkowski spacetime, and ω is the conformal factor relating the two metrics (i.e., $\gamma_{ab} = \omega^2 \eta_{ab}$). The boundary metric (i.e., rotating Einstein universe) is conformal to Minkowski space, since rotating Einstein universe after a change of variables is conformal to Einstein universe[6, 41], which is known to be conformal to Minkowski space. Notice that upon thermalizing the Brown–Cassidy expression, the coefficients α and δ will not depend on β (i.e., the temperature), since the same coefficients appear in the trace which is non–thermal.[4] Notice that the first term on the right hand side of the above expression vanishes at zero temperature if the manifold is conformally related to the whole Minkowiski space[38].

One way to understand the above expression is to think of it as the quantum version of the classical stress-energy tensor transformation law under conformal transformation $\gamma_{ab} = \omega^2 \eta_{ab}$.

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[4]The reason is that, the boundary metric does not depend on the mass parameter since the mass term vanishes at the boundary. But trace anomaly is a local geometric expression which depends only on the metric and its derivatives, hence it will not depend on β.

If a classical action is invariant under conformal transformation, then the stress-energy tensor will transform as

$$T_a{}^b(\gamma) = \omega^{-4}T_a{}^b(\eta) \tag{25}$$

provided that $T_a{}^b(\eta)$ is conserved. The existence of the last two terms in the right hand side of the expression is due to pure quantum effects (one loop corrections). This follows from realizing that they are the source of both the Casimir energy and the trace anomalies, which are pure quantum effects.

In ref.[12] we argued that in the limit $M \to 0$ the two stress–energy tensors on both sides of the duality are identical. The reason is that the field theory stress-energy tensor is protected from higher loop corrections as we mentioned earlier. Furthermore, and as we also already mentioned, once we have matched the conformal anomaly, it determines the entire tensor (at zero temperature) in the case we have here that the spacetime the field theory is defined on is conformally flat[38].

Now let us try to probe another aspect of this duality by going to another limit —the high temperature limit— checking the relation between the stress-energy tensors at weak and strong coupling (the latter defined by the AdS computation). At high $T = \beta^{-1}$, we can ignore quantum corrections in the field theory, since these quantum corrections do not depend on temperature, while the Minkowiski stress-energy tensor (the first term in the right hand side of equation (24)) goes as T^4. So in this limit we have:

$$<T_a{}^b(\gamma)>_{(\text{FT})} = \omega^{-4} <T_a{}^b(\eta)> \tag{26}$$

The previous relation is just the transformation law of stress-energy tensor under conformal transformation for a conformally invariant effective action. $<T_a{}^b(\eta)>$ is traceless and has the form

$$<T_a{}^b(\eta)> = \frac{\rho^{\text{M}}}{3}\left[4u_a u^b + \delta_a{}^b\right] \tag{27}$$

where the density is:

$$\rho^{\text{M}} = \frac{\pi^2 N^2}{2\beta_{\text{M}}{}^4} \,, \tag{28}$$

(the label "M", is for "Minkowski"), and again, u_a is a time–like unit vector. The stress–energy tensor of our field theory is given by

$$<T_a{}^b(\gamma)>_{(\text{FT})} = \frac{\rho^{\gamma}}{3}\left[4u_a u^b + \delta_a{}^b\right] \,, \quad \text{where} \quad \rho^{\gamma} = \frac{\pi^2 N^2}{2\beta_{\gamma}{}^4} \,, \tag{29}$$

10

with the label "γ" for the spacetime of our field theory. Notice that density gets scaled under conformal transformation

$$\rho^\gamma = \omega^4 \rho^M. \tag{30}$$

Then the field theory stress–energy tensor after dropping the label γ is given by

$$<T_a{}^b(\gamma)>_{\text{(FT)}} = \frac{\pi^2 N^2}{6\beta^4} \left[4 u_a u^b + \delta_a{}^b\right] . \tag{31}$$

On the gravity side one can express the mass parameter of Kerr–AdS$_5$ black hole as a function of β. For small β we get

$$M = \frac{\pi^4 l^6}{2G\beta^4} . \tag{32}$$

Substituting this value of M in the expression of stress–energy tensor in equation (9), and using equation (19), the stress–energy tensor on the gravity side is

$$<T_a{}^b(\gamma)>_{\text{(AdS)}} = \frac{\pi^2 N^2}{8\beta^4} \left[4 u_a u^b + \delta_a{}^b\right] \tag{33}$$

The two expressions for the stress-energy tensors are the same up to the usual 3/4 factor, which is now familiar ratio between energetic quantities from the strong and weak coupling regime of the theory[17].

The above argument can be used for rotating solutions with charge, as well with spherical or flat horizons, since after a change of coordinates, the boundary metric for these cases are also conformal to the whole Minkowiski space. In these cases we have temperature dependence similar to the Kerr–AdS case, because the mass as a function of temperature, contains other terms that depend on charges but which can be ignored at high temperature. The boundary metric of rotating charged black holes will be the same as the one for rotating black hole case, since terms that contain charges will vanish at the boundary. We expect that one can do the same for black holes with hyperbolic horizons. The only difference is that hyperbolic black holes boundaries are conformal to Rindler space, but this can be chosen as a part of Minkowiski space.

4   The Kerr–AdS$_6$ Solution

The general Kerr–AdS$_6$ has two rotation parameters, but we will consider only one parameter solution. The metric for Kerr–AdS$_6$ with one rotation parameter is given by

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + r^2\cos^2\theta d\psi^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\rho^2}{\Delta_r}dr^2 \\
&\quad + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi}d\phi\right)^2 + r^2\cos^2\theta d\Omega_2^2,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
\rho &= r^2 + a^2 cos^2(\theta) , \quad \Xi = 1 - a^2/l^2, \\
\Delta_r &= (r^2 + a^2)(1 + r^2/l^2) - 2MG/r , \\
\Delta_\theta &= 1 - a^2/l^2 \cos^2(\theta).
\end{aligned}
\tag{35}
$$

$d\Omega_2^2$ is the metric on the two–sphere

$$
d\Omega_2^2 = d\psi^2 + \sin^2 \psi d\eta^2.
\tag{36}
$$

The inverse temperature is given by

$$
\beta = \frac{4\pi(r_+^2 + a^2)r_+}{5r_+^4/l^2 + 3r_+^2(1 + a^2/l^2) + a^2} ,
\tag{37}
$$

the angular velocity has the form

$$
\Omega = a\frac{\Xi}{r_+^2 + a^2} .
\tag{38}
$$

Here, r_+ is the location of the horizon, the largest root of Δ_r.

The non–vanishing components of the stress–energy tensor at large r are given by,

$$
\begin{aligned}
T_{tt} &= \frac{M}{lG2\pi r^3} + O\left(\frac{1}{r^4}\right) , \\
T_{t\phi} &= -\frac{Ma\sin^2\theta}{lG2\pi\Xi r^3} + O\left(\frac{1}{r^4}\right) , \\
T_{\phi\phi} &= \frac{Ml}{\Xi^2 8\pi G r^3}(4\Xi - 5\Delta_\theta) + O\left(\frac{1}{r^4}\right) , \\
T_{\theta\theta} &= \frac{Ml}{\Delta_\theta 8\pi G r^3} + O\left(\frac{1}{r^4}\right) , \\
T_{\psi\psi} &= \frac{l\cos^2\theta M}{8\pi G r^3} + O\left(\frac{1}{r^4}\right) \\
T_{\eta\eta} &= \frac{\sin^2\theta \cos^2\theta lM}{8\pi G r^3} + O\left(\frac{1}{r^4}\right)
\end{aligned}
\tag{39}
$$

Using the definition for a conserved charge, one can calculate the mass and angular momentum of the solution:

$$
\mathcal{M} = \frac{4\pi}{3\Xi}M , \quad \mathcal{J} = \frac{2\pi}{3\Xi^2}aM .
\tag{40}
$$

Also the action is

$$
I_6 = -\frac{4\pi^2(r_+^2 + a^2)r_+\left[r_+^5/l^2 + r_+^3/l^2a^2 - MG\right]}{\Xi G\left(5r_+^4/l^2 + 3r_+^2(1 + a^2/l^2) + a^2\right)} .
\tag{41}
$$

The area of the horizon is given by

$$\mathcal{A} = 8\pi^2 G \frac{r_+^2(r_+^2 + a^2)}{3\Xi} \ . \tag{42}$$

The above quantities satisfy the thermodynamical relation (15) for $n = 5$.

The boundary metric is given by

$$ds^2 = \left[-dt^2 + \frac{2a\sin^2\theta}{\Xi} dt d\phi + l^2 \frac{d\theta^2}{\Delta_\theta} + l^2 \frac{\sin^2\theta}{\Xi} d\phi^2 + l^2 \cos^2\theta d\Omega_2{}^2 \right] \ . \tag{43}$$

Defining the tress tensor of the dual field theory by removing the conformal factor, one can put it in the compact form:

$$T_{ab} = \frac{M}{2\pi l^4} \left[5u_a u_b + \gamma_{ab} \right] \tag{44}$$

where $u^a = (1, 0, 0, 0, 0)$ and γ_{ab} is the boundary metric. It is interesting to notice that the tensor can be put in the standard form even in the presence of non–vanishing rotation parameter. We noticed this for Kerr–AdS$_4$ in ref.[12]. Conformal invariance together with conservation of the stress–energy tensor seems to constrain the form of the tensor. This property is likely shared by all odd dimensional field theory stress–energy tensor of this sort.

5 The Kerr–AdS$_7$ Solution

The general solution for Kerr–AdS$_7$ contains three rotation parameters, but we are going to consider only one parameter solution. The metric for the Kerr–AdS$_7$ is

$$\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a\sin^2\theta}{\Xi} d\phi \right)^2 + r^2 \cos^2\theta d\psi^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\rho^2}{\Delta_r} dr^2 + \\
&\quad \frac{\Delta_\theta \sin^2\theta}{\rho^2} \left(adt - \frac{(r^2 + a^2)}{\Xi} d\phi \right)^2 + r^2 \cos^2\theta d\Omega_3^2,
\end{aligned} \tag{45}$$

where the functions in the metric are the same except

$$\Delta_r = (r^2 + a^2)(1 + r^2/l^2) - 2MG/r^2 \ , \tag{46}$$

and $d\Omega_3^2$ is the metric on a three–sphere

$$d\Omega_3^2 = d\psi^2 + \sin^2\psi d\eta^2 + \cos^2\psi d\beta^2 \ . \tag{47}$$

The inverse temperature is

$$\beta = \frac{2\pi(r_+^2 + a^2)r_+}{3r_+^4/l^2 + 2r_+^2(1 + a^2/l^2) + a^2} \ , \tag{48}$$

and the angular velocity is the same as in the previous section. Once again r_+ is the location of the horizon, the largest root of Δ_r. The non–vanishing components for the stress–energy tensor at large r are given by,

$$
\begin{aligned}
T_{tt} &= \frac{l^3}{640\pi G r^4}\Big[(1 + a^2/l^2)(-131a^2/l^2 - 423(a^4/l^4)\cos^4\theta) + 219(1 + a^4/l^4)(a^2/l^2)\cos^2\theta \\
&\quad + 25(1 + a^6/l^6) + 235(a^6/l^6)\cos^6\theta + 400MG/l^4 + 492(a^4/l^4)\cos^2\theta\Big] + O(r^{-5})\ , \\
T_{t\phi} &= -\frac{l^3 a\sin^2\theta}{640\pi G r^4 \Xi}\Big[-231(a^6/l^6)\cos^4\theta + 5 + a^2/l^2 + 55(a^6/l^6)\cos^6\theta - 101a^4/l^4 + 400MG/l^4 \\
&\quad + 189(a^6/l^6)\cos^2\theta - 25a^6/l^6 + 168(a^4/l^4)\cos^2\theta - 51(a^4/l^4)\cos^4\theta - 3(a^2/l^2)\cos^2\theta\Big] + O(r^{-5}), \\
T_{\phi\phi} &= \frac{a l^5 \sin^2\theta}{640\pi G r^4 \Xi^2}\Big[102a^4/l^4 + 51(a^4/l^4)\cos^4\theta - 55(a^6/l^6)\cos^6\theta\,\Xi - 171(a^4/l^4)\cos^2\theta + 80MG/l^4 \\
&\quad + 189(a^8/l^8)\cos^2\theta + 3(a^2/l^2)\cos^2\theta + 4a^2/l^2 - 231(a^8/l^8)\cos^4\theta + 180(a^6/l^6)\cos^4\theta - 5 \\
&\quad + 480MG(a^2/l^6)\cos^2\theta - 21(a^6/l^6)\cos^2\theta - 76a^6/l^6 - 25a^8/l^8 + 400(a^2/l^6)MG\Big] + O r^{-5}), \\
T_{\theta\theta} &= -\frac{l^5}{640\pi G \Delta_\theta r^4}\Big[5(a^2/l^2)\Xi - 80MG/l^4 + 3(a^2/l^2)\cos^2\theta(1 + a^4/l^4) \\
&\quad + 66(a^4/l^4)\cos^2\theta - 45(a^4/l^4)\cos^4\theta(1 + a^2/l^2) + 25(a^6/l^6)\cos^6\Big] + O(r^{-5})\ , \\
T_{\psi\psi} &= -\frac{l^5\cos^2\theta}{640\pi G r^4}\Big[5\Xi(1 - a^4/l^4) - 80MG/l^4 - 51(a^4/l^4)\cos^4\theta(1 + a^2/l^2) \\
&\quad - 3(a^2/l^2)\cos^2\theta(1 + a^4/l^4) + 46(a^4/l^4)\cos^2\theta\Big] + O(r^{-5}), \\
T_{\eta\eta} &= \sin^2\psi\, T_{\psi\psi}, \\
T_{\beta\beta} &= \cos^2\psi\, T_{\psi\psi}.
\end{aligned}
\tag{49}
$$

The mass and angular momentum of the solution are:

$$
\mathcal{M} = -\frac{\pi^2 l^4\, [a^6/l^6 + 5a^4/l^4 + 50\Xi - 800MG/l^4]}{1280 G \Xi},\quad \mathcal{J} = \frac{\pi^2}{4\Xi^2} aM\ .
\tag{50}
$$

The energy dual to the Casimir energy of the field theory is given by setting $M = 0$ in the above, to give:

$$
\mathcal{E} = -\frac{\pi^2 l^4\, [a^6/l^6 + 5a^4/l^4 + 50\Xi]}{1280\Xi G}\ .
\tag{51}
$$

The action is

$$
\begin{aligned}
I_7 &= -\frac{\pi^3(r_+^2 + a^2)r_+ l^4}{640\Xi G\, (3r_+^4/l^2 + 2r_+^2(1 + a^2/l^2) + a^2)} \times \\
&\qquad\qquad \Big[160(r_+^6/l^6 + r_+^4/l^6 a^2 - MG/l^4) + 5a^4/l^4 + 50\Xi + a^6/l^6\Big]\ .
\end{aligned}
\tag{52}
$$

The area of the horizon is

$$
\mathcal{A} = \pi^3 G\frac{r_+^3(r_+^2 + a^2)}{\Xi}\ .
\tag{53}
$$

The above quantities satisfy the thermodynamical relation (15) for $n = 6$.

5.1 The Conformal Anomaly

The field theory to which the AdS_7 theory is supposed to be dual is[1] the "(0,2)" supersymmetric tensor field theory at large N. (See ref.[42] for a review.)

The metric on which the dual field theory resides is given using the procedure of section 2 by

$$ds^2 = \left[-dt^2 + \frac{2a \sin^2 \theta}{\Xi} dt d\phi + l^2 \frac{d\theta^2}{\Delta_\theta} + l^2 \frac{\sin^2 \theta}{\Xi} d\phi^2 + l^2 \cos^2 \theta d\Omega_3{}^2 \right] , \tag{54}$$

a higher dimensional generalization of the rotating Einstein universe. Taking the trace of the stress–energy tensor given in the previous section yields, after taking the limits in section 2:

$$\begin{aligned} \widehat{T}_a^a &= -\frac{a^2 N^3}{2\pi^3 l^8} \Big[5a^4/l^4 \cos^6 \theta - 8 \cos^4 \theta a^2/l^2 (1 + a^2/l^2) - 2(1 + a^2/l^2) \\ &\quad + 3 \cos^2 \theta (1 + a^4/l^4 + 3a^2/l^2) \Big] \end{aligned} \tag{55}$$

where we used the relation[1] $N^3 = 3\pi^2 l^5/16G$ between the field theory and the gravitational parameters.

Just as we saw for the case of Kerr–AdS_5, we see that this trace is a total derivative. Furthermore, we note that it can be written in terms of the Euler density:

$$\widehat{T}_a^a = -\frac{N^3}{4508\pi^3} E_6 . \tag{56}$$

The Euler density E_6 is displayed in *e.g.* ref.[16], where the field theory is discussed at weak coupling. The coefficient matches the results in ref.[21, 26, 16]. Just as in the four dimensional case, we see that for this special situation, the Euler density can be written in terms of type D quantities. In the notation of ref.[16], it is of the form $\sum_{i=1}^{7} d_i C_i$, with d_5 and d_7 zero, since they depend on the Weyl tensor, and $\{d_1, d_2, d_3, d_4, d_6\} = \{0, 1/9, 1/72, -5/12, 1/72\}$. (This is a useful parameterization, but of course, not unique.)

The conformal anomaly of the free field theory has been calculated in ref.[16] and the results are similar to the anomaly we computed on the AdS side, up to a factor of 4/7. The discrepancy between the two results is due to the fact that the coefficient in front of the Euler density in six dimensions is not controlled by a non–renormalization theorem as in the four–dimensional case. Consequently this coefficient will be different from the weakly coupled theory which may explain the 4/7 factor. Unfortunately there is no Casimir energy calculation for the free theory, and so we cannot compare it with our results. Nevertheless, we expect that the Casimir energy for the free theory will be different from the one calculated on the gravity side by exactly the same factor, since the conformal anomaly and the Casimir energy depend on the same coefficient, c_0, defined in equation (20).

6   Concluding Remarks

In closing, we note that overall, as stated in the introduction for other AdS black holes, the properties of the action and stress tensor of the Kerr–AdS solution in various dimensions seem to be consistent with a dual interpretation as those of a field theory in one dimension fewer. This is a quite satisfyingly successful examination of aspects of the AdS/CFT correspondence. This work is also an extensive demonstration of the usefulness of the counterterm subtraction method for computing intrinsic properties of quite complicated asymptotically AdS spacetimes.

It would be interesting to study the full thermodynamic phase structure of these higher dimensional rotating black holes, as done in four dimensions in ref.[8]. We expect that the properties of AdS will endow these thermodynamics with the physical properties required by the duality, in contrast to the $\Lambda = 0$ case.

While the dual field theories to Kerr–AdS reside on rotating spacetimes —generalizations of the rotating Einstein universe— which are situations of less immediate concern than, *e.g.*, gauge field theory at high density (to which the Reissner–Nordstrom black holes seem to be relevant), it is nevertheless an interesting arena in which to study the AdS/CFT correspondence and its generalizations.

We expect that there is more to be learned by further study of these systems. Particularly interesting would be further information from the pertubrative field theory side in the Kerr–AdS$_7$ case: The Casimir energy of the (0,2) tensor multiplet on the (rotating) six dimensional Einstein universe is an example of a quantity which would be nice to compare with the result we have presented here.

An obvious extension to study, for completeness, is the inclusion of all of the rotation parameters for AdS$_6$ and AdS$_7$, and comparison to the corresponding field theory quantities. As the two–parameter Kerr–AdS$_5$ case and the one parameter Kerr–AdS$_7$ case were both rather computationally intensive, we have not yet made serious attempts to progress beyond that level of complexity. We hope that progress on this can be made, in order to complete the story.

Acknowledgements

References

[1] J. Maldacena, Adv. Theor. Math. Phys. **2** 231 (1998), hep-th/9711200.

[2] E. Witten, Adv. Theor. Math. Phys. **2** 253 (1998), hep-th/9802150.

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B428** 105 (1998), hep-th/9802109.

[4] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983)

[5] A. Chamblin, R. Emparan, C. V. Johnson and R.C. Myers, Phys. Rev. **D60**, 064018 (1999), hep-th/9902170; A. Chamblin, R. Emparan, C. V. Johnson and R.C. Myers, Phys. Rev. **D60**, 104026 (1999), hep-th/9904197;

[6] S. W. Hawking, C. J. Hunter and M. M. Taylor–Robinson, Phys. Rev. **D59** (1999) 064005, hep-th/9811056.

[7] D. S. Berman and M. K. Parikh, Phys. Lett. **B463**, 168 (1999), hep-th/9907003.

[8] M. Caldarelli, G. Cognola, and D. Klemm, hep-th/9908022.

[9] R. B. Mann, Phys.Rev. **D60** (1999) 104047, hep-th/9903229.

[10] D. Klemm, JHEP **9811**, 019 (1998), hep-th/9811126.

[11] S. W. Hawking and H. S. Reall, Phys. Rev. **D61**, 024014 (2000), hep-th/9908109.

[12] A. M. Awad and C. V. Johnson, Phys. Rev. **D61** (2000) 084025, hep-th/9910040.

[13] J. Ho, hep-th/0005250.

[14] K. Landsteiner and E. Lopez, JHEP **9912** 020 (1999), hep-th/9911124.

[15] J. Bardeen and G. Horowitz, Phys. Rev. **D60** (1999) 104030, hep-th/9905099.

[16] F. Bastianelli, S. Frolov and A. A. Tseytlin, JHEP **0002**, 013 (2000), hep-th/0001041.

[17] S. S. Gubser, I. R. Klebanov and A. W. Peet, Phys. Rev. **D54** (1996) 3915, hep-th/9602065.

[18] A. M. Awad and C. V. Johnson, hep-th/0006037.

[19] S. Das and R. B. Mann, hep-th/0008028.

[20] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. **13**, 1487 (1996) gr-qc/9501014.

[21] M. Henningson and K. Skenderis, J.H.E.P. 9807 023 (1998), hep-th/9806087.

[22] G. Chalmers and K. Schalm, Phys. Rev. bf D61 046001 (2000), hep-th/9901144.

[23] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999), hep-th/9902121.

[24] R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. **D60**, 104001 (1999), hep-th/9903238.

[25] S. R. Lau, Phys. Rev. **D60** (1999) 104034, gr-qc/9903038.

[26] P. Kraus, F. Larsen and R. Siebelink, Nucl. Phys. **B563**, 259 (1999), hep-th/9906127.

[27] S. N. Solodukhin, hep-th/9909197.

[28] J. D. Brown and J. W. York, Phys. Rev. **D47**, 1407 (1993).

[29] R.C. Myers, hep-th/9903203.

[30] G. T. Horowitz and R. C. Myers, Phys. Rev. **D59**, 026005 (1999) hep-th/9808079.

[31] S. de Haro, K. Skenderis and S. N. Soloduklin, hep-th/0002230.

[32] S.S. Gubser and I.R. Klebanov, Phys. Lett. **413**, 41 (1997); D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, Nucl. Phys. **B526**, 543 (1998).

[33] D. Anselmi and A Kehagias, hep-th/9812092.

[34] S. Nojiri and S. Odintsov, Int. J. Mod. Phys. **A15**, 413 (2000), hep-th/9903033.

[35] M. Blau, K.S. Narian and E. Gava, hep-th/9904179.

[36] A. Bilal and C. Chu, Nucl. Phys. **B562** 181 (1999), hep-th/9907106; A. Bilal and C. Chu, hep-th/0003129.

[37] J. Brown and N. Cassidy, Phys.Rev. **D15** (1977) 2810.

[38] N. D. Birrell and P. C. W. Davies, *"Quantum fields in curved space"*, Cambridge University Press (1982).

[39] G. W. Gibbons and S. W. Hawking, Phys. Rev **D15**, 2752 (1977).

[40] S. Deser and A. Schwimmer, Phys. Lett. **B309** (1993) 279, hep-th/9302047.

[41] M.J. Cassidy and S.W. Hawking, Phys. Rev. **D57**, 2372 (1998) hep-th/9709066.

[42] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.